

DENSITY  
OF  
SPHERE PACKINGS

Jeff Lagarias

AT&T Labs-Research

Florham Park, NJ

## Topics

1. Packings in Euclidean Space  
 $\mathbb{R}^n$
2. Density of Sphere Packings
3. Packings in Hyperbolic Space  
 $\mathbb{H}^n$ .  
(Bouen-Radin)

---

Survey of recent <sup>exciting</sup> results.  
(None of them due to speaker.)

# 1. Sphere - Packings in Euclidean Space

---

$\mathbb{R}^n$

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n)\}$$

$$\|\underline{x}\|^2 = x_1^2 + x_2^2 + \dots + x_n^2$$

$$\langle \underline{x}, \underline{y} \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

real scalar product

---

Volume of ball of radius r

$$\text{Vol}(B_n(r)) = e_n r^n$$

where

$$e_n = \text{Vol}(B_n(1)) = \frac{\pi^{n/2}}{r(\frac{1}{2})} \sim n^{-\frac{n}{2}} c^n$$

rapidly goes  
to 0  
as  $n$   
increases

# Lattices

$L \subseteq \mathbb{R}^n$  discrete additive subgroup

$$L = \mathbb{Z} b_1 + \mathbb{Z} b_2 + \cdots + \mathbb{Z} b_n$$

$[b_1, \dots, b_n]$  linearly independent over  $\mathbb{R}$

called basis of  $L$ .

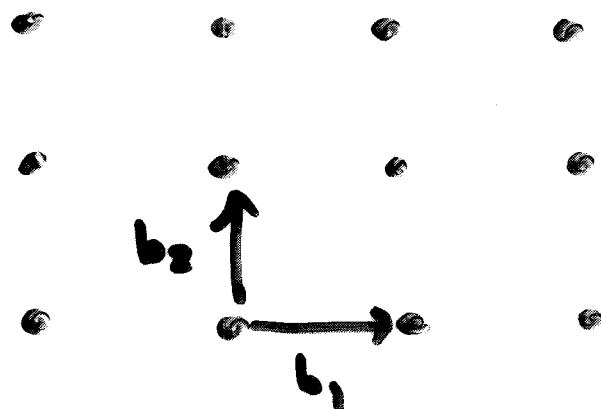
These are full rank lattices  $\Leftrightarrow$

$\mathbb{R}^n/L$  is compact.

$$\text{Vol}(\mathbb{R}^n/L) = \det(L)$$

$$:= \left| \det \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \right|$$

$L = \mathbb{Z}^n$  integer lattice



$$\underline{b}_1 = (1, 0)$$
$$\underline{b}_2 = (0, 1)$$

$\mathbb{Z}^2$



$$\underline{b}_1 = (1, 3)$$
$$\underline{b}_2 = (1, 4)$$

$\mathbb{Z}^2$

## Lattice Sphere-Packings

Spheres are placed at lattice points

$$\begin{aligned} \text{Maximum radius allowed} &= \frac{1}{2} \lambda_1(L) \\ &= \frac{1}{2} (\text{length shortest vector of } L) \end{aligned}$$

Density of a lattice packing  $\Delta(L)$

$$\begin{aligned} \Delta(L) &= \frac{\text{"area covered by spheres" }}{\text{total area}} \\ &= \frac{e_n \left( \frac{1}{2} \lambda_1(L) \right)^n}{\det(L)} \\ &= \frac{\pi^{n/2} (\lambda_1(L))^n}{2^n \Gamma(n/2) \det(L)} \end{aligned}$$

If  $\lambda_1(L) = 1$ , then

$$\Delta(L) = \frac{\text{Vol}(B_n)}{2^n \text{Vol}(\mathbb{R}^n/\mathbb{Z})}$$

# DUAL LATTICE      ( Reciprocal Lattice )

$$L^* = \left\{ b^* \in \mathbb{R}^n : \langle b, b^* \rangle \in \mathbb{Z} \text{ for all } b \in L \right\}$$


---

Dual basis :  $B^* = [b_1^*, \dots, b_n^*]$

$$\langle b_i, b_j^* \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$$


---

$$\det(L^*) = \frac{1}{\det(L)}$$

# Lattice $E_8$

in  $\mathbb{R}^8$

①  $\det(E_8) = 1$

②  $(x_1, \dots, x_8) \in E_8$

if each  $x_i$  is  $\begin{cases} \text{integer} \\ \text{or} \\ \text{half-integer} \end{cases}$

and

$$x_1 + x_2 + \dots + x_8 \equiv 0 \pmod{2}.$$

③  $E_8$  is self-dual.

$$E_8 = (E_8)^*$$

Leech Lattice  $\Lambda_{24}$  in  $\mathbb{R}^{24}$

$\det(\Lambda_{24}) = 1$ , self-dual, even

$[\langle x, w \rangle \equiv 0 \pmod{2}]$ , shortest vector of length 4.

## Sphere-Packing Density

For arbitrary discrete set  $\Lambda$  (not necessarily lattice)  
 the sphere-packing density of  $\Lambda$  is

$\Delta(\Lambda) =$  "fraction of space  $\mathbb{R}^n$   
 covered by maximal  
 packing radius spheres  
 centered at points of  $\Lambda"$

$\Delta(n) =$  maximal density over  
 all discrete sets  $\Lambda$   
 $=$   $n$ -dim sphere-packing density

$\Delta_{\text{Lat}}(n)$  = maximal density over all  
lattice packings (using a lattice  $L$ )

(May require  $\det(L) = 1$ )  
 without loss of generality

# Absolute Density

## (1) Upper Density

$$\bar{d}(\mathcal{P}; T) = \sup_{x \in \mathbb{R}^n}$$

$$\frac{\text{Vol}(\mathcal{P} \cap B(x; T))}{\text{Vol}(B(x; T))}$$

$$\bar{d}(\mathcal{P}) = \limsup_{T \rightarrow \infty} \bar{d}(\mathcal{P}; T)$$

## (2) Lower Density

$$\underline{d}(\mathcal{P}, T) = \inf_{x \in \mathbb{R}^n} \frac{\text{Vol}(\mathcal{P} \cap B(x; T))}{\text{Vol}(B(x; T))}$$

$$\underline{d}(\mathcal{P}) = \liminf_{T \rightarrow \infty} \underline{d}(\mathcal{P}; T)$$

(3) Defn. A packing  $\mathcal{P}$  has an absolute density if

$$\bar{d}(\mathcal{P}) = \underline{d}(\mathcal{P})$$

Call it  $\Delta(\mathcal{P})$ .

# Sphere-Packing Density in $\mathbb{R}^n$

## Two Important Points

### (1) Scale-Invariance

The density of packing by equal-radius spheres  
doesn't depend on the radius  $r$ .

Homothety:  $\underline{x} \rightarrow 2\underline{x}$  on  $\mathbb{R}^n$

changes radius  $r \rightarrow 2r$

but leaves density unchanged.

### (2) Large-Ball Limit

As sphere radius  $T \rightarrow \infty$ , the ratio

$$\frac{\text{Surface Area } (B(0;T))}{\text{Vol } (B(0,T))} \sim O\left(\frac{T^{n-1}}{T^n}\right) \rightarrow 0$$

as  $T \rightarrow \infty$ .

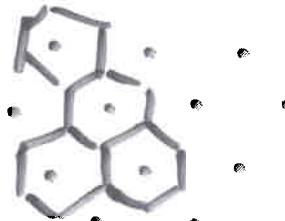
"Surface-Area" effects are negligible in limit.

# DENSEST SPHERE PACKING

---

Known in dims 1, 2, 3.

dim 1  $\mathbb{Z}$



dim 2  $A_2$  = Hexagonal packing

dim 3  $A_3$  = "Cassiniell packing"

(Kepler's Conjecture)

[T. Hales 1998-2003<sup>+</sup>]

[part of Hilbert's 18<sup>th</sup> problem]

---

These are all lattice packings.

---

Unknown in dims  $\geq 4$ .

# BOUNDS ON OPTIMAL SPHERE PACKING

Theorem. In  $n$  dimensions, optimal sphere packing density has:

$$2^{-n} \leq \Delta_{\text{Lat}}(n) \leq \Delta(n) \leq 2^{-0.599n}$$

Lower Bound: Minkowski ~1900

- "random lattice" works.

Upper Bound: Kabatiansky & Levenshtain 1979

- 2-pt homogeneous spaces

The  
Big Problem.

① Find the optimal exponent  $\alpha$  in:

$$\Delta(n) \leq 2^{-\alpha n}$$

as  $n \rightarrow \infty$ . [Improve known bounds]

---

Know:  $0.599 \leq \alpha \leq 1$ .

---

"We don't understand how things fit in Euclidean space  $\mathbb{R}^n$  for large  $n$ ."

---

② Find the optimal exponent  $\alpha_L$  in

$$\Delta_{Lat}(n) \leq 2^{-\alpha_L n}$$

as  $n \rightarrow \infty$ .

[Is  $\alpha = \alpha_L$ ?]

## LATTICE PACKING PROBLEM

Do there exist dimensions  $n$  in which the densest sphere-packing of  $\mathbb{R}^n$  is not a lattice packing?

- Folklore: This ought to be true in all large dimensions  $n > n_0$ .
- Densest known packings in dims 10, 11 are non-lattice packings.  
( Densest known packings in dims  $\leq 9$  are lattice packings )

# Regularity of Densest Packings

Working

Conjecture : In each dimension  $n \geq 1$ ,

there exists a periodic packing

that is a densest packing.

Periodic packing :

(Lattice) + (finite # of  
sets of L)

"crystalline packing"

- There is no good reason to believe this conjecture holds in high dimensions.

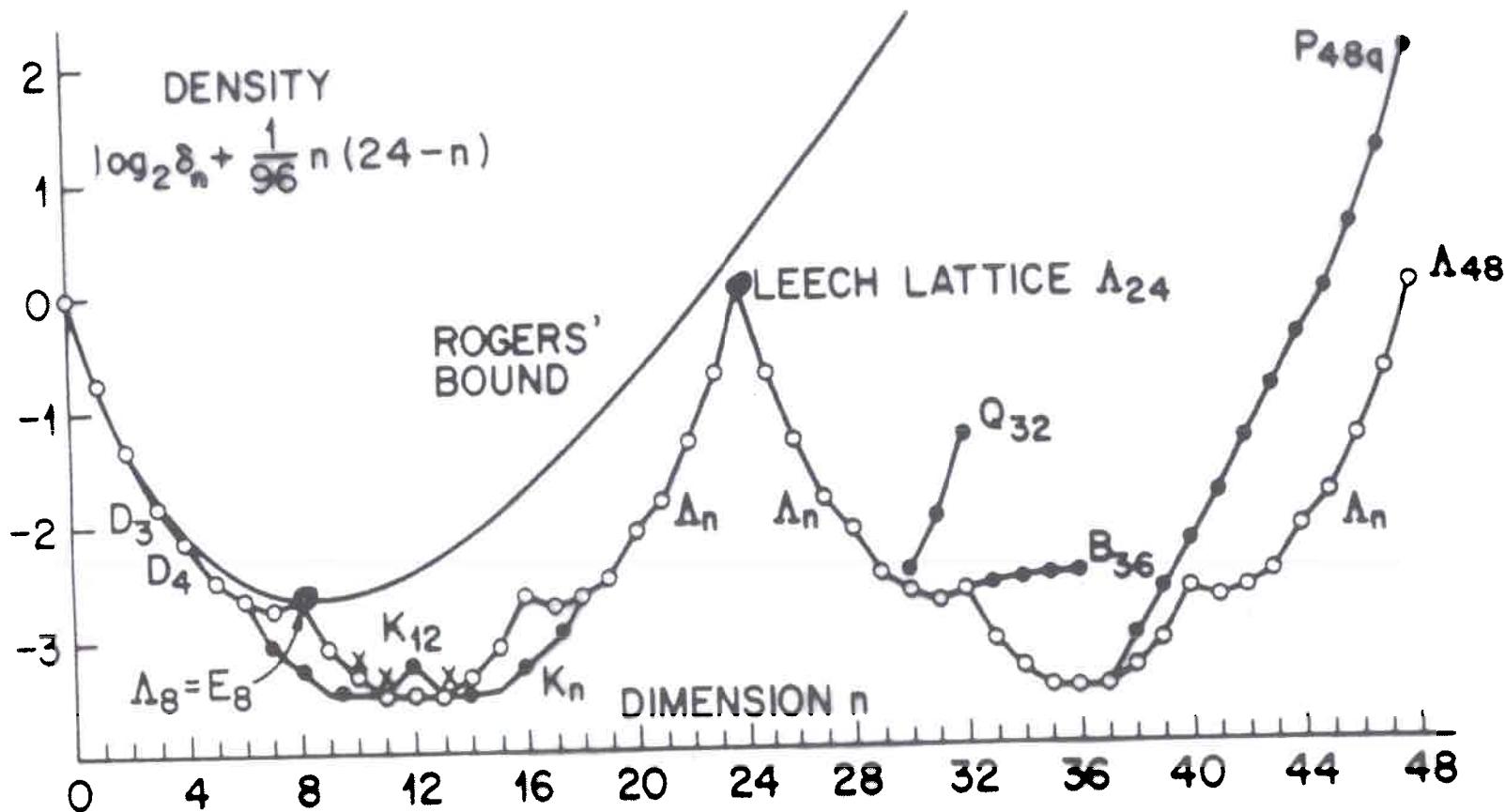


Figure 1.5. The densest sphere packings known in dimensions  $n \leq 48$ .  
The vertical axis gives  $\log_2 \delta + n(24-n)/96$ , where  $\delta$  is the center density.

$$\delta_n = \frac{\Delta(n)}{\text{Vol}(B_n)}$$

Σ center density

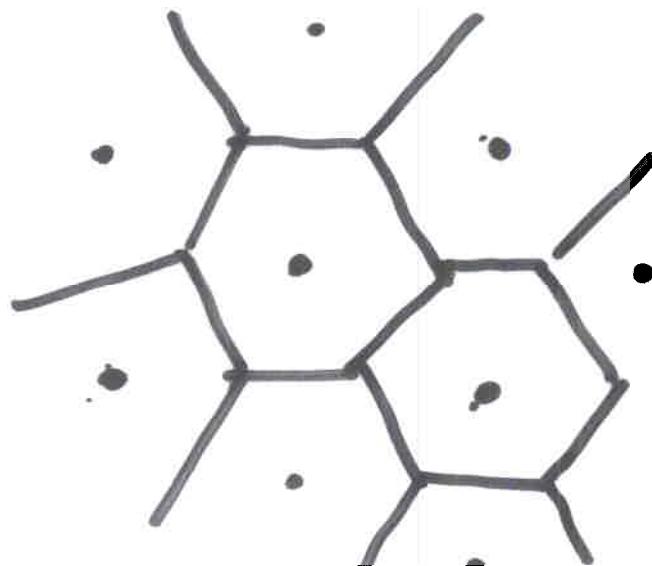
Ref: Conway & Sloane  
Sphere Packings, Lattices and Groups

# Low Dimensional Packings

# Dimensions 2. ( L. Fejes-Tóth ) 1940

Dimension 3. ( T. Hales  
1998-2003+ )

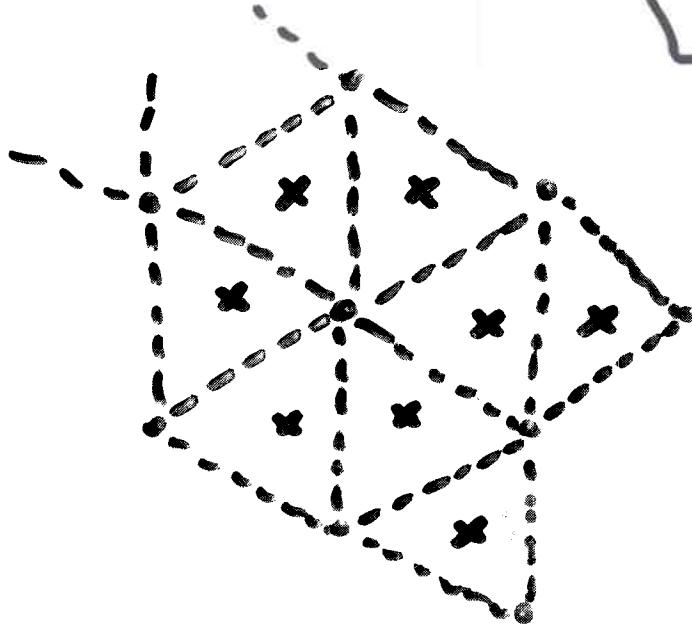
# VORONOI & DELAUNAY TILINGS



Voronoi Cell  
 $= \{ \text{nearest points to sphere center} \}$

Voronoi cells

$\Leftrightarrow$  sphere center



Delaunay tiling

$= \{ \text{connect centers of adjacent Voronoi cells} \}$

Delaunay Regions

$\Rightarrow$  vertices of Voronoi cells

- "Generic" Delaunay cells are simplices.

# DELAUNAY      SIMPLEX BOUND

---

Theorem. In  $\mathbb{R}^n$ ,

$$\Delta(n) \leq \max$$

Delaunay  
Simplex  $C$

[saturated  
packings  
only]

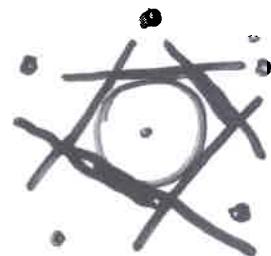
$$\left( \frac{\text{Vol}(B_n) \cap C}{\text{Vol}(\text{Delaunay Simplex } C)} \right)$$

at vertices

## Remarks

- Local bound
- Tight in dim 2.
- Not tight in dims  $\geq 3$ .
- Conjectured to match Rogers' bound;  
i.e. Optimum is regular simplex  
of side length 2. (dim 23)

# Voronoi Band



Theorem. In  $\mathbb{R}^n$ ,

$$\Delta(n) \leq \max_{\substack{\text{Voronoi} \\ \text{cells} \\ V}} \left( \frac{\text{Vol}(B_n)}{\text{Vol}(\text{Voronoi cell})} \right)$$

## Remarks

- Local bound
- Tight in dim 2 (gives proof)
- Not tight in dim 3
- Possibly tight in  
dimensions 4, 8, 24.

# Dimension 3.

(Hales 1999-2003+)

[see JCL paper: DCG 2002]

## Local density approach

Assign weighted sum of volume near given sphere center to it ( $\frac{\text{distance}}{\text{radius}} < 12\sqrt{2}$ ) so that all space partitioned among sphere centers, with weight 1. Prove

$$\left( \begin{array}{l} \text{Local volume} \\ \text{assigned each} \\ \text{center} \end{array} \right) \geq \sqrt{32}$$

If so

$\Rightarrow$

$$\left( \begin{array}{l} \text{Local cell} \\ \text{area} \\ \text{covered by} \\ \text{sphere} \end{array} \right) \geq \frac{\pi}{\sqrt{18}}.$$

Kepler bound

T. Hales & S. Ferguson local density partition

uses combination of

Voronoi-type ; Delaunay-type regions at each vertex.

# Fourier Analysis Method

(Henry Cohen  
 Noam Elkies)  
 Annals 2003

Theorem. [Cohen &  
ElKies]

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be any  
 "admissible" function such that

$$(1) \quad f(\underline{x}) \leq 0 \quad \text{for } \|\underline{x}\| \geq 1.$$

$$(2) \quad \hat{f}(\underline{\xi}) \geq 0 \quad \text{for all } \underline{\xi} \in (\mathbb{R}^n)^*.$$

Then

$$\Delta(n) \leq \frac{\text{Vol}(B_n)}{2^n} \cdot \frac{f(0)}{\hat{f}(0)}.$$

Notes: ① Fourier transform

$$\hat{f}(\underline{\xi}) = \int_{\mathbb{R}^n} f(\underline{x}) e^{2\pi i(x_1 \xi_1 + \dots + x_n \xi_n)} dx_1 dx_2 \dots dx_n$$

② "Admissible" means Poisson summation formula valid:

Sufficient Condition:  $|f(\underline{x})| \leq (1 + \|\underline{x}\|)^{-n-\varepsilon}$  ;  $|\hat{f}(\underline{\xi})| \leq (1 + \|\underline{\xi}\|)^{-n-\varepsilon}$

# Fourier Analysis Method - 2

Theorem. [Cohn & Elkies]  
2003

In dimension 8,

$$1 \leq \frac{\Delta(8)}{\Delta(E_8)} \leq 1.00000001$$

$E_8$ -lattice

In dimension 24,

$$1 \leq \frac{\Delta(24)}{\Delta(\Lambda_{24})} \leq 1.00001$$

Leech lattice

Proof.

Find suitable function  $f(\underline{x})$   
by linear programming method.  $\square$

## Fourier Analysis Method - 3

Can the Fourier analysis method  
prove  $E_8$  and Leech lattice  $\Lambda_{24}$   
 are extremal in dims. 8 and 24?

- Need a "magic function"  $f(\underline{x})$ .

- Can reduce to radial functions

$$f(r) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}.$$

- An extremal radial function must have

$$f(r) = 0 \text{ when } r \in \left\{ \begin{array}{l} \text{length of any} \\ \text{vector in } \Lambda \end{array} \right\}$$

$$\hat{f}(\xi) = 0 \text{ when } \xi \in \left\{ \begin{array}{l} \text{length of any} \\ \text{vector in } \Lambda^* \end{array} \right\}$$

Here  $\Lambda = E_8 \text{ or } \Lambda_{24}$

$$\Lambda = \Lambda^* \quad \frac{\text{self-dual}}{\text{lattice}}$$

- Henry Cohn dual LP approach.

## 2. Density of Sphere-Packings

### in General Spaces

Test Case :  $H^n$  = <sup>n-diml</sup>  
<sup>hyperbolic</sup>  
<sup>space.</sup>

Sphere-Packings are considered

- in Grassmannians  $G(n, d)$
- in general Riemannian manifolds
- in finite spaces (coding theory)
  - (Hamming distance)
  - (Lee distance etc.)

### Non-compact Spaces

lead to trouble...

## Density Method 0. ( Absolute Density )

Define

$$\bar{d}(P, T) = \sup \left( \begin{array}{c} \text{density in} \\ \text{all balls of} \\ \text{size } T \end{array} \right)$$

$$\underline{d}(P, T) = \inf \left( \begin{array}{c} \text{density in} \\ \text{all balls of} \\ \text{size } T \end{array} \right)$$

Set

$$\bar{d}(P) = \limsup_{T \rightarrow \infty} \bar{d}(P, T)$$

$$\underline{d}(P) = \liminf_{T \rightarrow \infty} \underline{d}(P, T)$$

Say  $P$  has absolute density if

$$\bar{d}(P) = \underline{d}(P)$$

Call it  $\Delta(P)$  or  $\Delta_{\text{abs}}(P)$ .

# Density Method 1.

( Limit Density on Finite Regions )  
expanding around "center."

(1) Pick "center"  $\underline{o}$ .

Compute density in ~~sphere~~<sup>ball</sup> of radius  $T$   
around  $\underline{o}$

$$d_T(P) = \frac{\text{Vol}(P \cap B(\underline{o}; T))}{\text{Vol}(B(\underline{o}; T))}$$

(2) Set upper density

$$\bar{d}(P) = \limsup_{T \rightarrow \infty} d_T(P)$$

(3) Set lower density

$$\underline{d}(P) = \liminf_{T \rightarrow \infty} d_T(P)$$

(4) Say packing  $P$  has a <sup>(central)</sup> density  $d$

if  $\bar{d}(P) = \underline{d}(P) = d$

for all centers  $\underline{x}_0$ .

## Density Method 2.

(Limit of "Periodic" Packings)

① Take a compact space

$$M_p := \mathbb{R}^n / \Gamma \cup \mathbb{H}^n / \Gamma$$

$\Gamma$  = co-compact group of isometries

Pack  $M_p$  with equal radius spheres.

② Take limit as  $\text{Vol}(M_{\Gamma_m}) \rightarrow \infty$ ,  
letting  $\Gamma_m$  vary,  $m \rightarrow \infty$ .

③ Packing on  $M_{\Gamma_m}$  lifts to "periodic"  
packing on  $\mathbb{R}^n$  (resp.  $\mathbb{H}^n$ )

Periodic packings  $P_{\Gamma_m}$  go to  
limit packing  $P$ .

# Density Method 3.

[Bauer-Rudin]  
[O. Schramm]

(Ergodic Theory)

Move the packing around by  
isometries. Compute

density( $P$ ) := Prob  $\left[ \begin{array}{l} \text{origin } \underline{O} \text{ is} \\ \text{covered by a} \\ \text{ball in } P \end{array} \right]$

## Remark.

Here the packing  $P$  is used to  
(sometimes)  
construct:

probability measure  $\mu = \mu_P$

on a "space of packings" with  
balls of radius  $r$ .

Then  $\text{density}(P) =$  "average density"  
for probability  
measure  $\mu$ .

## Density Methods: Summary

① Method 0 used classically  
on  $\mathbb{R}^n$ .

It was into ~~some~~ trouble on  $\mathbb{H}^n$ .

② Method 3 leads to a  
working answer on  $\mathbb{H}^n$ . (Bauer &  
Radin)

It also gives answers in  
spirit of Method 1 and Method 2.

It may give extra insight  
in case of  $\mathbb{R}^n$ ; suggests  
aperiodic extremal packings might  
occur in high dimensions.

### 3. Sphere-Packings in Hyperbolic

Space  $\mathbb{H}^n$ .

$\mathbb{H}^n =$  Constant negative curvature  
space

#### Upper Half-Space Model

$$(x_1, x_2, \dots, x_{n-1}, y) : y > 0$$

$$ds^2 = \frac{1}{y^2} (dx_1^2 + \dots + dx_{n-1}^2) \quad \text{line element}$$

$$dV = \frac{dx_1 dx_2 \dots dx_{n-1} dy}{y^n} \quad \text{volume element}$$

#### Volume of ball of radius r

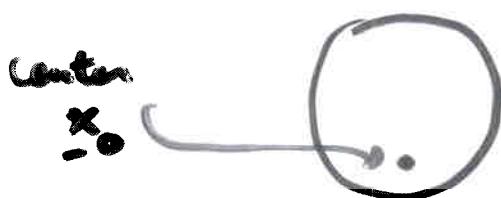
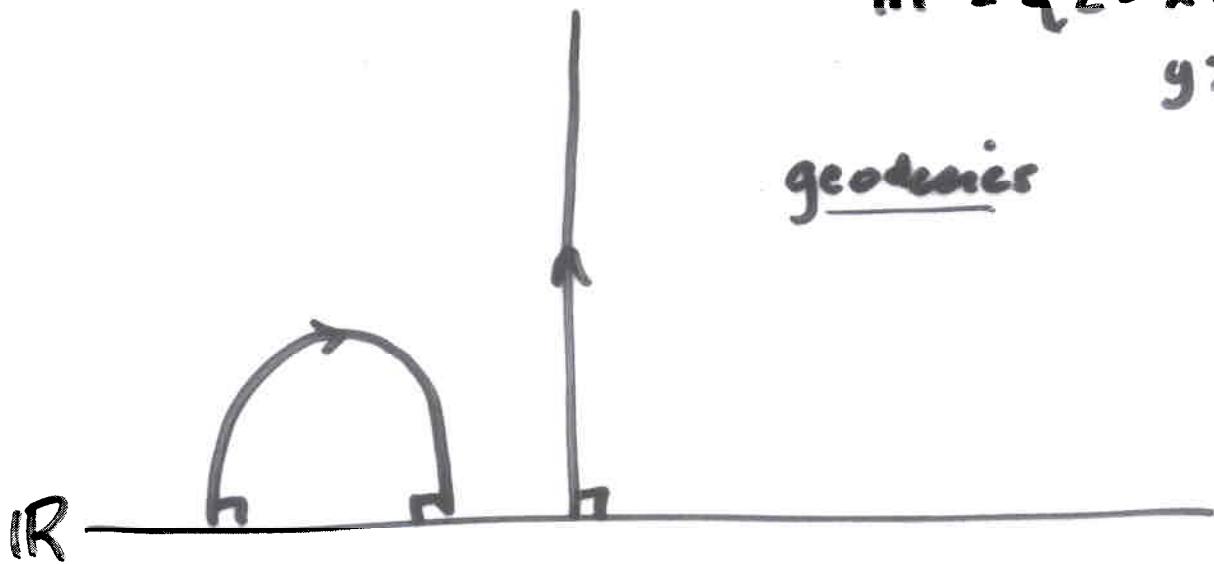
$$\text{Vol}(B_n(r)) = e_{n-1} \int_0^{\infty} (\sinh x)^{n-1} dx$$

$$\text{Vol}(B_2(r)) = \pi (e^r + e^{-r})$$

# Upper Half-Plane Model of $\mathbb{H}^2$

$$\mathbb{H}^2 = \{ z = x + yi \mid y > 0 \}$$

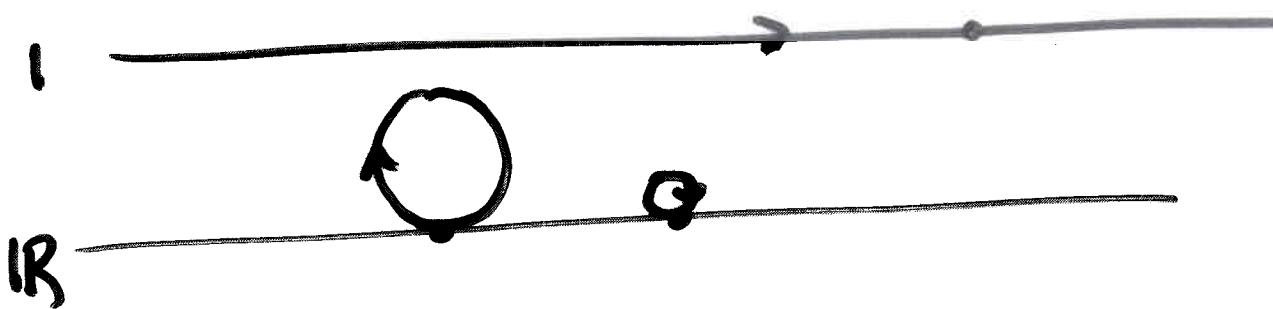
geodesics



$|x - x_0| = r$   
 "hyperbolic" spheres =  
 Euclidean circles  
 (centers  
 are  
 different)



horocycles =  
 spheres of  
 infinite radius



# Isometry Group of Hyperbolic Plane $\mathbb{H}^2$

$$\text{Iso}(\mathbb{H}^2) = \text{PSL}(2, \mathbb{R})$$

$$= \text{SL}(2, \mathbb{R}) / \pm I$$

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad g \cdot z = \frac{az + b}{cz + d}$$

linear fractional transformation

$g$  maps geodesics to geodesics

(hyperbolic) circles to circles

horocycles to horocycles

$\Rightarrow \text{Iso}(\mathbb{H}^2)$  acts on

{set of circle packings} 

# Sphere Packing Densities in Hyperbolic Space?

FACT. The maximal density  
attainable by a sphere-packing in  $H^n$   
depends on the radius of  
the spheres used.

---

Homothety.  $\underline{x} \rightarrow \lambda \underline{x}$  in Euclidean  
space changes radius of sphere,  
leaves sphere-packing density unchanged.

Homothety unavailable in  $H^n$ .

(The map is a hyperbolic isometry, leaving  
radius unchanged.)

# HYPERBOLIC SPHERE-PACKING DENSITIES?

Problem. Sphere-Packing densities

don't exist for some "reasonable"  
packings in  $\mathbb{H}^n$ .

Origin of Problem. As  $T \rightarrow \infty$ ,

$$\text{Volume} \left( \begin{array}{c} \text{large ball} \\ \text{of} \\ \text{radius } T \end{array} \right) \approx \frac{\text{Surface Area}}{\text{Area}} \left( \begin{array}{c} \text{large ball} \\ \text{of} \\ \text{radius } T \end{array} \right)$$

$\Rightarrow$  "Surface Area effects"

cannot be ignored as  $T \rightarrow \infty$ .

Böröczky example in  $\mathbb{H}^2$ .

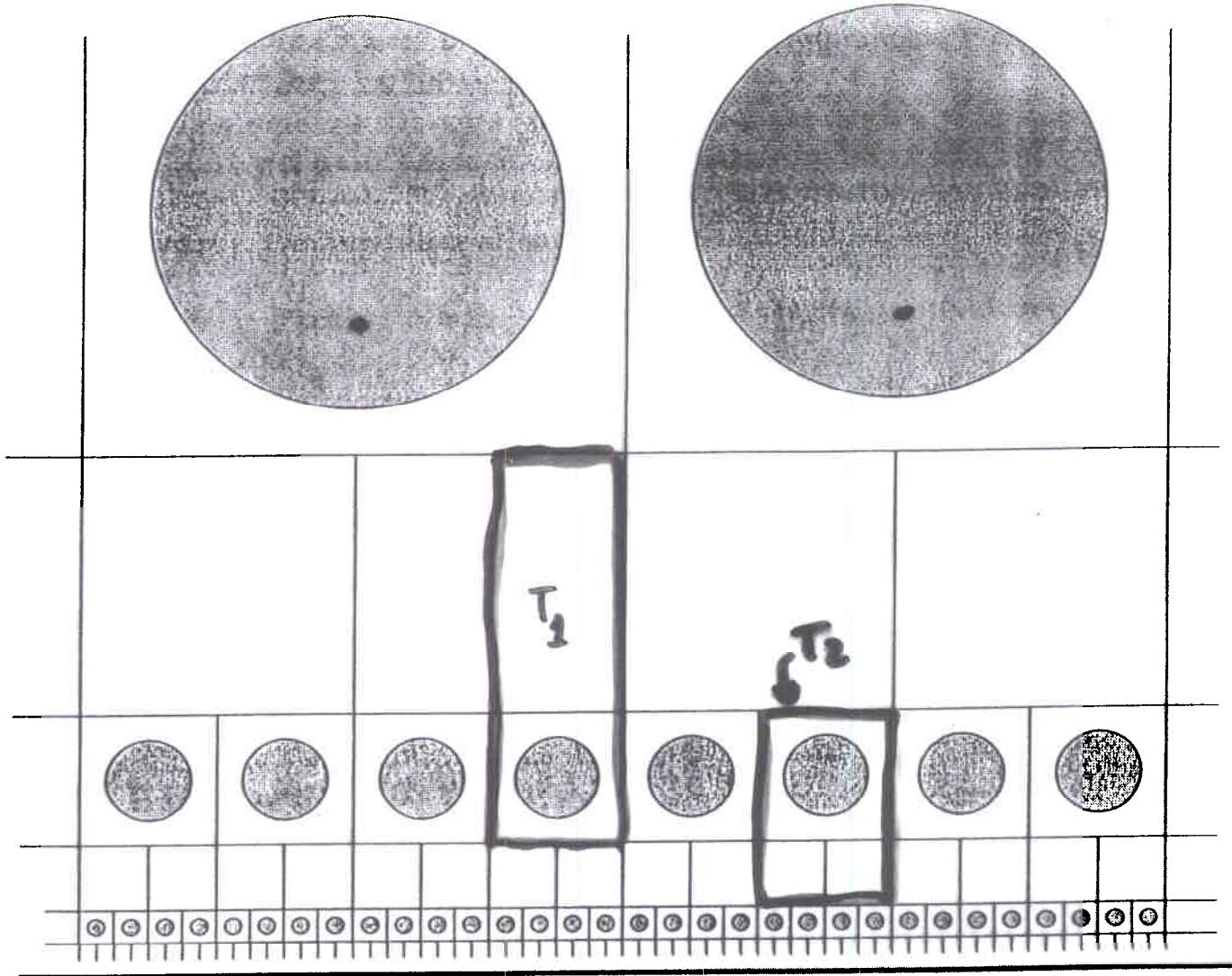


Figure 1. Boroczky's packing of disks

$$\text{Vol}(T_2) = 2 \text{ Vol}(T_1)$$

$$\text{Tiling Set} = \left\{ \begin{bmatrix} 1^m \\ 0 \\ 1 \end{bmatrix} : m \in \mathbb{Z} \right\} \times \left\{ \begin{bmatrix} 2^k \\ 0 \\ 2^{-k} \end{bmatrix} : k \in \mathbb{Z} \right\}$$

$\curvearrowleft$

## Bowen - Radin Approach

Although one cannot define "density of packing" for many packings, one can define a notion of  
 "densest packing".

## Ergodic theory approach:

- ① A "packing" is identified with the closure of its orbit under hyperbolic motions in a "space of packings":  
 $\Rightarrow$  Family of "locally indistinguishable packings"
- ② Can define "average density" of packing using (ergodic) invariant probability measure on orbit, if such measure exists.
- ③ Maximize average density over all packings. measures on spaces

Example.

$\text{PSL}(2, \mathbb{R})$  acts on compact space

$$\partial \mathbb{H}^2 = \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$$

by linear fractional transformations

$$g \cdot z = \begin{cases} \frac{az+b}{cz+d} & z \neq -d/c \\ \infty & z = -d/c \end{cases}$$


---

FACT. There is no invariant

Borel probability measure on  $\mathbb{P}^1(\mathbb{R})$   
for this action.

## SATURATED PACKINGS

---

A Saturated Packing is one to which no additional sphere can be inserted.

---

A Completely Saturated Packing is a saturated packing that is "locally optimal" in sense that one cannot remove  $k$  spheres and then insert back  $k+1$  spheres, for any finite  $k \geq 1$ .

---

FACT. Saturated packings have Voronoi cells of bounded radius.

# Spaces of Sphere - Packings

Defn.  $\delta_n(r) = \left\{ \begin{array}{l} \text{all saturated packings} \\ \text{of } \mathbb{H}^n \text{ with} \\ \text{radius } r \text{ spheres} \end{array} \right\}$

Lemma. The space  $\delta_n(r)$  has a topology with which it is a compact

space and

$g \in \text{Iso}(\mathbb{H}^n)$ ,  $P \in \delta_n(r)$   
acts on it so that the map

$$(g, P) \longmapsto g(P)$$

is jointly continuous.

## Measures on $\mathcal{S}_n(r)$

$$\mathcal{M}(r) = \left\{ \begin{array}{l} \text{Borel measures } \mu \text{ on } \mathcal{S}_n(r) \\ \text{having finite mass 1} \\ \text{= probability measures} \end{array} \right\}$$

UI

$$\mathcal{M}_I(r) = \left\{ \begin{array}{l} \mu \text{ invariant under} \\ \text{Iso}(\mathbb{H}^n) \text{ in sense} \\ \mu(E) = \mu(g.E) \end{array} \right\}$$

UI

$$\mathcal{M}_I^e(r) = \left\{ \begin{array}{l} \text{ergodic } \text{Iso}(\mathbb{H}^n) \\ \text{invariant probability measures} \end{array} \right\}$$

UI

$$\mathcal{M}_I^{ue}(r) = \left\{ \begin{array}{l} \text{uniquely ergodic } \text{Iso}(\mathbb{H}^n) \\ \text{invariant probability measures} \end{array} \right\}$$

- All of these classes are nonempty.

3.11

# "AVERAGE" DENSITY OF INVARIANT MEASURE $\mu$

Defn. If invariant measure  $\mu \in M_I(r)$ ,

let

$$\chi_0(\rho) = \begin{cases} 1 & \text{if } \underline{0} \text{ inside a ball of packing } \rho \\ 0 & \text{otherwise} \end{cases}$$

Then  $\chi_0 : S_n(r) \rightarrow \mathbb{R}_+$  is a Borel function on  $S_n(r)$  and the average density of  $\mu$  is:

$$D(\mu) := \frac{1}{S_n(r)} \int \chi_0(\rho) d\mu(\rho)$$

Remark.  $\underline{0}$  used as "center" of packing but result independent of choice of center by  $\text{Iso}(\mathbb{H}^n)$ -invariance.

# OPTIMAL DENSITY MEASURES

---

Defn. For radius  $r$  spheres in hyperbolic space  $\mathbb{H}^n$ , set optimal density

$$\Delta_n(r) := \sup_{\mu \in M_I^e(r)} D(\mu).$$


---

Theorem (Babenko-Rudin 2003) For  $\mathbb{H}^n$ , any  $r > 0$ ,

there exists an ergodic invariant measure  $\mu$  attaining the optimal density

$$\Delta_n(r) = D(\mu).$$

Here

$$\mu \in M_I^e(r).$$

# "OPTIMAL" PACKINGS

## THEOREM ("Optimal packings")

(1) [Bauer-Radin] There is a full  $\mu$ -measure set of packings  $\rho$  that attain the density  $\Delta_n(r)$  in the sense of

$$\Delta_n(r) = \frac{\text{average Voronoi cell density of } \rho}{\text{average Delaunay cell density of } \rho}$$

$$\Delta_n(r) = \text{average Delaunay cell density of } \rho$$

$\Delta_n(r) = \text{limit density expanding around a }\underset{\text{sphere}}{\text{center}} \underline{\Omega}.$

(2) [Bauer] There is a full  $\mu$ -measure set of packings that are completely saturated.

(3) [Bauer] In dim 2, for  $H^2$  the optimal density is a limit of "periodic packings" (any <sup>sphere</sup> radius  $r$ ).

# "Periodic" Packings in $H^n$

---

These packings have symmetry group  
 $\Gamma \subseteq \text{Iso}(H^n)$  with  $H^n/\Gamma$  co-compact

$\Gamma \subseteq \text{Iso}(H^n)$  with  $H^n/\Gamma$  compact.

The orbit

$$\text{Iso}(H^n)(P) \subseteq S_n(r)$$

has invariant measure  $\mu$  induced  
 from

$$\text{Iso}(H^n)(P) = (\text{Iso}(H^n)/\Gamma)(P)$$

$$\downarrow \text{ "is"} \\ \text{Iso}(H^n/\Gamma) \leftarrow \text{finite volume.}$$

Then

$$\begin{aligned} D(\mu) &= \text{Voronoi density } (P) \\ &= \text{Delaunay density } (P) \end{aligned}$$

# OPTIMAL HYPERBOLIC PACKINGS &

## APERIODICITY

Theorem. [Boyer-Roddin 2003]

For almost all "radii"  $r$   
 the optimal density  $\Delta_{\mathbb{H}^n}(r)$   
 is not attained by any  
 "periodic" packing.

Defn.

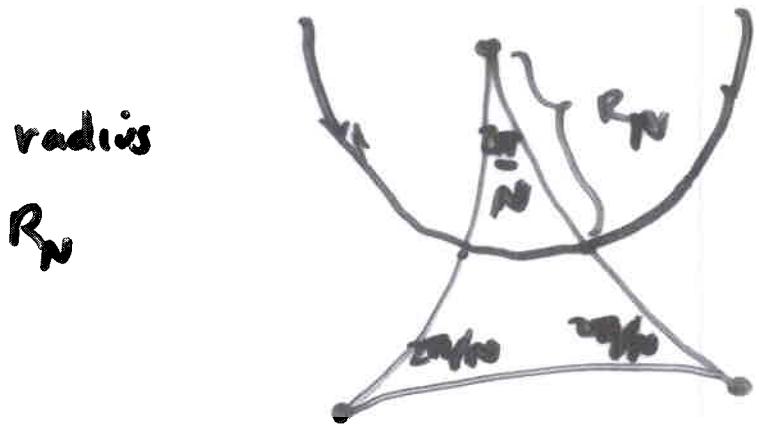
"periodic" packing  $P \Leftrightarrow$   
 $P$  has cocompact subgroup  $\Gamma$  of  
 $\text{Iso}(\mathbb{H}^n)$  as symmetries

$$g(P) = P, \quad g \in \Gamma$$

Some

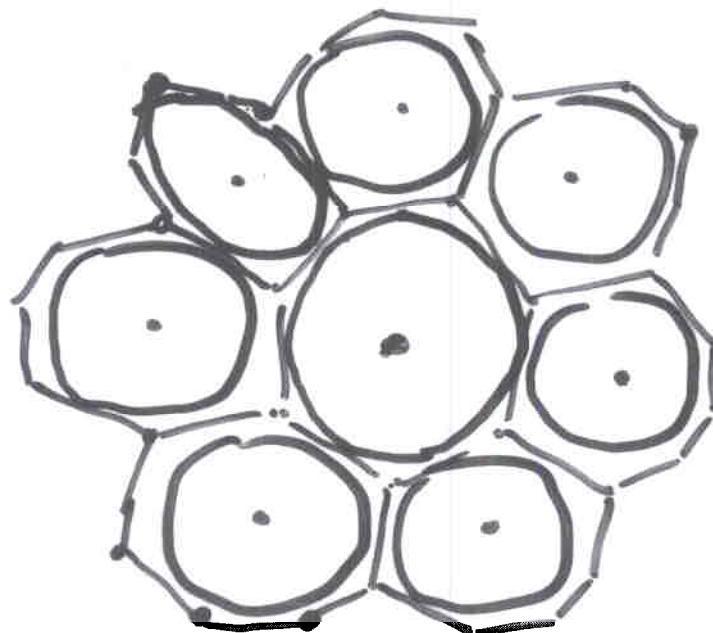
# "Perfect" Hyperbolic Packings

L. Fejes- Toth



$$N \geq 7$$

Tiling by  
 $(\frac{2\pi}{N}, \frac{2\pi}{N}, \frac{2\pi}{N})$   
 equi-angular  
 triangles  
 (Delonean)  
 simplices



Tiling of  
 $\mathbb{H}^2$  by  
 "regular"  
 heptagons  
 (Voronoi)  
 cells

$\Gamma_N$  = cocompact subgroup of  
 $\text{Iso}(\mathbb{H}^2)$ .

Voronoi cell = fundamental  
 domain  $\mathbb{H}^2/\Gamma_N$

# OPTIMAL DENSITIES IN $H^2$

## HYPERBOLIC PLANE

Radius R

0

Optimal Density	$D_2(R)$
-----------------	----------

$$\frac{\pi}{\sqrt{2}} = 0.9091 \leftarrow \text{Euclidean space}$$

$R_3$

0.9142

$$\frac{\frac{3}{\sin \frac{\pi}{n}} - 6}{n - 6}$$

$R_8$

0.9196

⋮

⋮

$R_\infty$

$$\frac{3}{\pi} = 0.9549$$

Ideal Hyperbolic Triangle

Thm. (Bauer 2003)

- $D_2(R)$  is continuous function of  $R$ . sphere radius

- $D_2(R)$  is not known to be

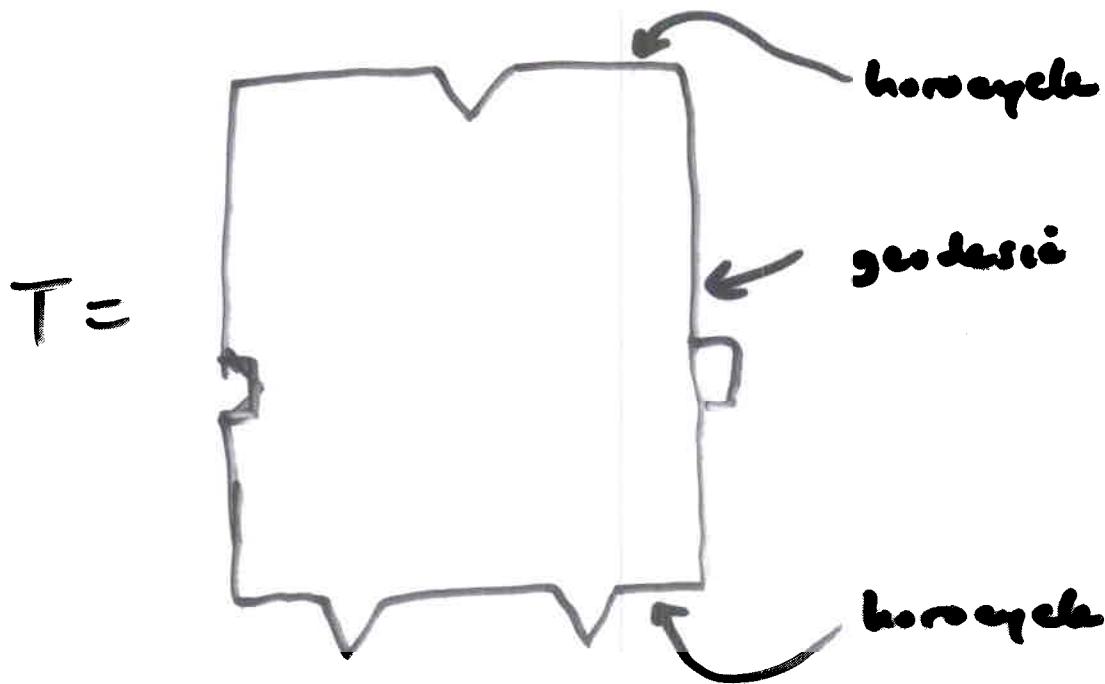
increasing in  $R$ , ~~and~~ might (not) be.

## UPPER BOUND ON DENSITY

Theorem. (Roth  
Keleti-Halli) 1998

The "density" of the horoball packing in  $H^n$  gives an upper bound on the "density"  $\bar{J}(R)$  of sphere packings by spheres of any radius  $R$  in  $R^n$ ,  $S^n$ ,  $H^n$ .

## PROBLEMS WITH OPTIMAL DENSITY



Theorem: (Bauer-Rudin)

(1) The region  $T$  tiles  $\mathbb{H}^2$

$$\Rightarrow \Delta_{\text{absolute}}(T) = 1.$$

(2) The optimal density has

$$\Delta_{\text{opt}}(T) < 1.$$

- This is variant of Bôröczky example.

## Conclusion

(1) Representation theory of the semi-simple Lie group  $\begin{matrix} PSL(2, \mathbb{R}) \\ \sqcup \\ Iso(\mathbb{H}^2) \end{matrix}$  is relevant to the analysis.

(2) Some actions of this group on a compact space have no invariant Borel measures.

(Only quasi-invariant measures.)



"non-commutative geometry"

(3) Packings & Tilings potentially will lead to interesting representation-theory questions for such spaces and their discrete subgroups  $\Gamma$ .